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LOCAL H-MAPS OF CLASSIFYING SPACES

BY
TIMOTHY LANCE[1]

ABSTRACT. Let BU denote the localization at an odd prime p of the classifying space for stable complex bundles, and let f: $BU \to BU$ be an H-map with fiber F. In this paper the Hopf algebra $H^*(F, \mathbf{Z}/p)$ is computed for any such f. For certain H-maps f of geometric interest the p-local cohomology of F is given by means of the Bockstein spectral sequence. A direct description of $H^*(F, \mathbf{Z}_{(p)})$ is also given for an important special case. Applications to the classifying spaces of surgery will appear later.

0. Introduction. This paper presents a computation of the cohomology of the fibers of maps f: $BU \to BU$, where BU denotes the localization at a fixed odd prime p of the classifying space for stable complex bundles, which preserve the H-multiplication induced by Whitney sum of bundles. Using the p-local H space equivalence $BU \simeq BO \times \Omega^2 BO$ a computation of the cohomology of the fibers of H-maps $BO \to BO$ is also obtained. In a subsequent paper these results will be applied to a detailed study of certain classifying spaces of surgery which arise in this fashion.

The viewpoint of this paper is motivated by a result of Adams that an H map f: $BU \to BU$ is determined up to homotopy by the induced homomorphism $f_\#$: $\pi_*(BU) \to \pi_*(BU)$. If we write $\mathbf{Z}_{(p)}$ for the integers localized at p and define the characteristic sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ of f by the condition that $f_\#$ is multiplication by $\lambda_j \in \mathbf{Z}_{(p)}$ on $\pi_{2j}(BU) = \mathbf{Z}_{(p)}$, then in principle the cohomology of the fiber can be completely described in terms of λ. We thus begin by studying the characteristic sequence, using as our main tool a natural decomposition of Hopf algebras $H^*(BU, \mathbf{Z}_{(p)}) \cong \bigotimes_{n \text{ prime to } p} A_n^*$ in which A_n^* is a polynomial Hopf algebra on generators $a_{n,k}^*$ of degree $2np^k$, $k = 0, 1, \ldots$.

THEOREM A. *If f: $BU \to BU$ is an H-map with characteristic sequence λ and $n_1 \equiv n_2 \bmod (p-1)$, then p^{j+1} divides $\lambda_{n_1 p^j} - \lambda_{n_2 p^j}$.*

Define the surplus sequence $s(f)$ of f by $s(f)_m = \nu(\lambda_m) - \nu(m)$ where $\nu(x)$ is the exponent of the highest power of p dividing x. An important step in the

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proof of Theorem A is the fact that if n is prime to p and j is minimal such that $s(f)_{np^j} \leqslant 0$, then $s(f)_{np^{j+k}} = -k$ for all $k \geqslant 0$. Thus for any n prime to p we may define $\delta_n = j$ if $s(f)_{np^j} = 0$ and $\delta_n = \infty$ if no such j exists. For any integer $\delta \geqslant 0$ let ξ^δ denote the Frobenius map $x \to x^{p^\delta}$, and let ξ^∞ be the augmentation.

THEOREM B. *Suppose f: $BU \to BU$ is an H-map with fiber F and index δ_n defined as above. There is an isomorphism of Hopf algebras*

$$H^*(F, \mathbf{Z}/p) \cong \bigotimes_{n \text{ prime to } p} E\{\sigma a^*_{n,j} | 0 \leqslant j < \delta_n\} \otimes (A^*_n // \xi^{\delta_n} A^*_n \otimes \mathbf{Z}/p)$$

in which σ: $H^q(BU, \mathbf{Z}/p) \to H^{q-1}(F, \mathbf{Z}/p)$ is the cohomology suspension. Moreover, the map induced on cohomology by $F \to BU$ is given by the natural projections $A^*_n \otimes \mathbf{Z}/p \to A^*_n // \xi^{\delta_n} A^*_n \otimes \mathbf{Z}/p$.

Perhaps the most transparent way to display the p-local cohomology is by means of the Bockstein spectral sequence. This is the graded E_1 spectral sequence associated to the exact couple given by the cohomology sequence of $0 \to \mathbf{Z} \to^p \mathbf{Z} \to \mathbf{Z}/p \to 0$. Of particular interest for geometric reasons is the Bockstein spectral sequence of the fiber of a "two-step function" f. This is an H-map such that $s(f)_n = s(f)_{pn} = \cdots = s(f)_{np^{\delta_n - 1}} = s_n > 0$ for each n prime to p such that $\delta_n > 0$.

THEOREM C. *If f: $BU \to BU$ is a two-step function with fiber F, then the E_r term of the Bockstein spectral sequence of F is given by*

$$\bigotimes_{n \text{ prime to } p} E\{\sigma a^*_{n,\overline{r-s}_n+j} | 0 \leqslant j < \delta_n\} \otimes (A^*_n(\overline{r-s}_n) // \xi^{\delta_n} A^*_n(\overline{r-s}_n) \otimes \mathbf{Z}/p)$$

where $\overline{r-s_n} = \max\{0, r - s_n\}$ *and*

$$A^*_n(k) = \mathbf{Z}_{(p)}[a^*_{n,k}, a^*_{n,k+1}, \ldots] \quad \textit{for any } k \geqslant 0.$$

In an application of this work to smoothing theory it will be necessary to have an explicit description of the $\mathbf{Z}_{(p)}$ cohomology of the fiber of an H-map f: $BU \to BU$ whose index δ_n never exceeds 1. Let S denote the set of all n prime to p such that $\delta_n = 1$, and write T for the set of all sequences $\alpha = (\alpha_1, \alpha_2, \ldots)$ of nonnegative integers each of whose nonzero entries is $\leqslant p - 1$ and lies in one of the subsequences $\alpha_n, \alpha_{np}, \ldots$ for some $n \in S$. For each $\alpha \in T$ write $l(\alpha)$ for the number of such subsequences containing a nonzero entry, and let $\operatorname{ind}(\alpha) = \min_{n \in S} p^{\nu(\lambda_n) + i_n}$ where $\alpha_{np^{i_n}}$ is the first nonzero entry of $\alpha_n, \alpha_{pn}, \ldots$ (if none exists, $i_n = \infty$). Let $w(\alpha) = \alpha_1 + 2\alpha_2 + 3\alpha_3 + \ldots$.

THEOREM D. *If f: $BU \to BU$ is an H-map with fiber F such that $\delta_n \leq 1$ for each n prime to p, then for each $m > 0$*

$$H^m(F, \mathbf{Z}_{(p)}) \cong \bigoplus_{j=1}^{\infty} \bigoplus_{i=0}^{\infty} \bigoplus_{\substack{\alpha \in T \\ l(\alpha)=j \\ 2w(\alpha)=m+i}} (\mathbf{Z}/\mathrm{ind}(\alpha))^{\binom{j-1}{i}}$$

where the right-hand term denotes the direct sum of $\binom{j-1}{i}$ copies of $\mathbf{Z}/\mathrm{ind}(\alpha)$.

The paper is arranged as follows. In §1 we discuss the arithmetic of characteristic sequences and the basic congruence of Theorem A. This depends on the decomposition $H^*(BU, \mathbf{Z}_{(p)}) \cong \bigotimes_{n \text{ prime to } p} A_n^*$ which is established in §2. §3 is devoted to a detailed study of the homomorphism f^*: $H^*(BU, \mathbf{Z}_{(p)}) \to H^*(BU, \mathbf{Z}_{(p)})$ induced by an H-map f. This permits us to calculate torsion products for various p local rings R which, by some remarkable collapse theorems for the Eilenberg-Moore spectral sequence, is equivalent to computing the R cohomology of the fiber of f. This is done for $R = \mathbf{Z}/p$ in §4 to obtain Theorem B and for $R = \mathbf{Z}_{(p)}$ in §5 to obtain Theorem C. In §6 we carry out the calculation Theorem D.

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1. The characteristic sequence. All spaces in this paper are assumed to be localized at a fixed odd prime p and thus have p-local homotopy groups and reduced homology groups ([25], [26]). Write μ for the localization at p of the H-multiplication (or an appropriate iterate) induced by Whitney sum of bundles; μ corresponds to loop multiplication under the Bott equivalence $BU \simeq \Omega_0^2 BU$. One immediate benefit of localizing is a geometric splitting of BU.

1.1 THEOREM (ADAMS-PETERSON). *There exist equivalences of infinite loop spaces*

$$BU \to W \times \Omega^2 W \times \cdots \times \Omega^{2p-4} W \quad \textit{and}$$
$$BO \to W \times \Omega^4 W \times \cdots \times \Omega^{2p-6} W$$

where $\pi_{2k(p-1)}(W) = \mathbf{Z}_{(p)}$, $k = 1, 2, \ldots$, and $\pi_i(W) = 0$ otherwise.

W may be defined as the bottom space of a spectrum associated to a bordism theory with singularities [19]. The H-space decomposition can be found in [19] or [1], and the infinite loop equivalence then follows from [2]. Notice in particular that $BU \simeq BO \times \Omega^2 BO$, so any H-map f: $BO \to BO$ may be regarded as a factor of the H-map $f \times 1$: $BU \to BU$ with the same fiber. Similarly, 1.1 and Bott periodicity yield an equivalence $BO \simeq BSp$.

For any map f: $BU \to BU$ we define a sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ in $\mathbf{Z}_{(p)}$ by the condition that $f_\#$ equals multiplication by λ_j on $\pi_{2j}(BU) = \mathbf{Z}_{(p)}$. If f is an

H-map (that is, if $f\mu = \mu(f \times f)$) we refer to λ as the *characteristic sequence of* f because of the following.

1.2 Lemma [**15**, p. 100]. *If two H-maps f, g: $BU \to BU$ have the same characteristic sequence they are homotopic.*

Using 1.1 we may define infinite loop maps $\Omega^{2i}W \to BU$ and $BU \to \Omega^{2j}W$ such that the composite $\Omega^{2i}W \to BU \to \Omega^{2j}W$ is the identity or constant depending on whether or not $i = j$. Using these the following decomposition of H-maps follows directly from 1.2.

1.3 Corollary. *For any H-map f: $BU \to BU$ let $\tilde{f}_{2k}$ denote the composite $\Omega^{2k}W \to BU \to^{f} BU \to \Omega^{2k}W$. Then the diagram below homotopy commutes.*

$$\begin{array}{ccc} BU & \xrightarrow{f} & BU \\ \simeq\downarrow & & \downarrow\simeq \\ \prod_{j=0}^{p-2} \Omega^{2j}W & \xrightarrow{\tilde{f}_0 \times \tilde{f}_2 \times \cdots \times \tilde{f}_{2p-4}} & \prod_{j=0}^{p-2} \Omega^{2j}W \end{array}$$

1.4 *Operations on characteristic sequences.* Given H-maps f, g: $BU \to BU$ with characteristic sequences λ and η we may form new H-maps $f + g = \mu \circ (f \times g) \circ \Delta$ and $f \circ g$ whose characteristic sequences are the pointwise sum $\lambda + \eta$ and product $\lambda \cdot \eta$, respectively. The loop space inverse -1 has characteristic sequence $(-1, -1, \dots)$, so that $-f = (-1) \circ f$ has sequence $-\lambda$. Adding copies of 1 or (-1) we obtain maps with characteristic sequences of any constant integer value n. These are homotopy invertible for n prime to p, and thus for any $\rho \in \mathbf{Z}_{(p)}$ there is an H-map with constant sequence ρ.

The splitting 1.3 induces a shearing operation. Let S be some subset of $\{0, 2, \dots, 2p - 4\}$ and let

$$\tilde{f}_s\colon \prod_{j=0}^{p-2} \Omega^{2j}W \to \prod_{j=0}^{p-2} \Omega^{2j}W$$

be the product whose $2j$th component is $\tilde{f}_{2j}$ if $2j \in S$ and 1 otherwise. By 1.3, $\tilde{f}_S$ induces a unique H-map f_S: $BU \to BU$ with characteristic sequence $\lambda(S)$ defined by $\lambda(S)_n = \lambda_n$ or 1 depending on whether or not $2n + 2j \equiv 0 \mod 2(p - 1)$ for some $2j \in S$. Setting $S = \{0, 4, \dots, 2p - 6\}$ we may thus identify an H-map f: $BO \to BO$ with f_S: $BU \to BU$ with characteristic sequence $(1, \lambda_2, 1, \lambda_4, \dots)$ and fiber(f) = fiber(f_S). We can also mix characteristic sequences. If S_1 and S_2 partition $\{0, 2, \dots, 2p - 4\}$, for example, then $f_{S_1} \circ g_{S_2} = g_{S_2} \circ f_{S_1}$ has characteristic sequence $\lambda(S_1) \cdot \eta(S_2)$, a mixture of λ and η. Finally, note that by Bott periodicity we have a loop map $\Omega^{2k}f$: $BU \to BU$ whose characteristic sequence $(\lambda_{k+1}, \lambda_{k+2}, \dots)$ is a left shift of λ.

1.5 DEFINITION. For any $\beta \in \mathbf{Z}_{(p)}$ let $\nu(\beta)$ denote the exponent of p in a prime power decomposition of the numerator of β (let $\nu(0) = \infty$). We say that $\beta_1 \equiv \beta_2 \bmod p^k$ if $\nu(\beta_1 - \beta_2) \geqslant k$. Define the *surplus sequence* $s(f)$ of an H-map $f\colon BU \to BU$ with characteristic sequence λ by $s(f)_n = \nu(\lambda_n) - \nu(n)$. A positive surplus sequence is necessary and sufficient for the triviality of f^*: $H^*(BU, \mathbf{Z}/p) \to H^*(BU, \mathbf{Z}/p)$ (3.7), and in general surplus measures the torsion in the cohomology of the fiber (compare §5). In §3 we will also establish the following.

1.6 LEMMA. *For any n prime to p, if j is minimal such that $s(f)_{np^j} \leqslant 0$, then $s(f)_{np^{j+k}} = -k$.*

1.7 THEOREM. *Let $f\colon BU \to BU$ be an H-map with characteristic sequence λ. Then $\lambda_{mp^k} \equiv \lambda_{np^k} \bmod p^{k+1}$ whenever $m \equiv n \bmod (p-1)$.*

PROOF. If we replace f by $g = f - \lambda_{np^k} \cdot 1$ as in 1.4, the resulting characteristic sequence has 0 as its np^kth term. In particular, $s(f)_{np^k} > 0$ and thus $s(f)_{np^j} > 0$ for $j < k$ by 1.6. But this means that p^{j+1} divides $\lambda_{np^j} - \lambda_{np^k}$ for $j < k$. Using these congruences it thus suffices to check that $\lambda_{np^j} \equiv \lambda_{(n+p-1)p^j} \bmod p^{j+1}$ for any n, j. But if $k = p^j(n-1)$, it follows by 1.4 that $\Omega^{2k}f$ has a characteristic sequence with p^jth term λ_{np^j} and p^{j+1}st term $\lambda_{(n+p-1)p^j}$. The desired congruence now follows from 1.6 as above. □

1.8 REMARK. Suppose $f\colon BU \to BU$ is any map (not necessarily an H-map) with characteristic sequence λ. Then $\Omega^{2j}f\colon BU \to BU$ is an H-map with characteristic sequence $(\lambda_{j+1}, \lambda_{j+2}, \ldots)$. Applying 1.7 for various values of j it follows easily that $\lambda_{mp^k} \equiv \lambda_{np^k} \bmod p^{k+1}$ if $m \equiv n \bmod (p-1)$ and $m \neq 1 \neq n$. I conjecture that the latter restriction on m and n is not necessary. But in any case, by the shearing construction of 1.4 we may identify any map f: $BO \to BO$ with a map $f_S\colon BU \to BU$ with characteristic sequence $(1, \lambda_2, 1, \lambda_4, \ldots)$ so that 1.7 holds for $\lambda_2, \lambda_4, \ldots$. These congruences may be viewed as a generalized Kummer congruence; in a subsequent paper we study a cannibalistic class $\rho\colon BO \to BO^{\otimes}$ for which 1.7 reduces to the classical congruences between Bernoulli numbers.

2. Homology of local classifying spaces. Let R be a commutative principal ideal domain which is p-local. This means that $R \cong R \otimes_{\mathbf{Z}} \mathbf{Z}_{(p)}$ as a group or, equivalently, that multiplication by m is bijective for m prime to p. We write $c_n \in H^{2n}(BU, R)$ for the class associated by localization to the nth Chern class, and note that cup product and the coproduct induced by the Whitney sum map μ give $H^*(BU, R)$ the following familiar Hopf algebra structure ([13], [16]).

2.1 Theorem. *$H^*(BU, R)$ is a polynomial Hopf algebra $R[c_1, c_2, \dots]$ with coproduct $\mu^* c_n = \Sigma c_i \otimes c_{n-i}$. If $d_n \in H_{2n}(BU, R)$ is dual (in the basis of monomials) to c_1^n, then the correspondence $c_n \to d_n$ defines an isomorphism of Hopf algebras $H^*(BU, R) \to H_*(BU, R)$.*

For any n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of weight $w(\alpha) = \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n$ write c^α for the cup product $c_1^{\alpha_1} \cdots c_n^{\alpha_n} \in H^{2w(\alpha)}(BU, R)$ with dual class $d_\alpha \in H_{2w(\alpha)}(BU, R)$, and let $d^\alpha = d_1^{\alpha_1} \cdots d_n^{\alpha_n}$ be the Pontrjagin product induced by μ_*. Going full circle, write c_α for the dual of d^α in the monomial basis in $H_*(BU, R)$ (the Chern class c_n is in fact dual to d_1^n). By 2.1 it follows that the primitives in $H^*(BU, R)$ and $H_*(BU, R)$ are generated as R modules by $\{c_{e_1}, c_{e_2}, \dots\}$ and $\{d_{e_1}, d_{e_2}, \dots\}$, respectively (where $e_n = (0, \dots, 0, 1)$ is the nth unit vector), and $c_{e_n} \to d_{e_n}$ under the isomorphism of 1.2. By [5] we may choose generators $\omega_n \in \pi_{2n}(BU) = \mathbf{Z}_{(p)}$ which are carried by the Hurewicz map to $(m-1)! d_{e_n}$. We can also describe the primitives directly. For any α let $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and

$$\{\alpha\} = (\alpha_1 + \cdots + \alpha_n)!/\alpha_1! \cdots \alpha_n!.$$

2.2 Theorem. $d_{e_n} = \Sigma_{w(\alpha)=n}(-1)^{|\alpha|+n} n\{\alpha\} d^\alpha / |\alpha|$.

An identical formula holds for c_{e_n}. If we identify c_n with the nth elementary symmetric function as in [4], then 2.2 is a consequence of Waring's formula for the Newton polynomials [13].

Given indeterminates $t_0, t_1, \dots$ define the kth Witt polynomial T_k by $T_k(t) = t_0^{p^k} + p t_1^{p^{k-1}} + \cdots + p^k t_k$ where we abbreviate $t = (t_0, t_1, \dots)$. Let $t^p = (t_0^p, t_1^p, \dots)$.

2.3 Lemma. *Let $P_0, P_1, \dots$ be polynomials in $t_0, t_1, \dots$ with coefficients in R (respectively, $\mathbf{Z}$) such that $P_k(t) - P_{k-1}(t^p)$ vanishes $\bmod p^k$. Then the equations $P_k(t) = T_k(\varphi_0(t), \varphi_1(t), \dots, \varphi_k(t))$ inductively define polynomials $\varphi_0, \varphi_1, \dots$ with coefficients in R (respectively, $\mathbf{Z}$).*

Proof. Suppose the polynomials P_i are integral, the proof for R being identical. Since $\varphi_0 = P_0$, we assume inductively that $\varphi_0, \dots, \varphi_{k-1}$ are integral for some $k > 0$. Since $T_k(t) = T_{k-1}(t^p) + p^k t_k$, we must verify that $P_k(t) - T_{k-1}((\varphi_0(t))^p, \dots, (\varphi_{k-1}(t))^p)$ vanishes $\bmod p^k$.

Evidently the integral polynomials $(\varphi_j(t))^p$ and $\varphi_j(t^p)$ are congruent $\bmod p$. Applying the binomial theorem inductively, it follows that $(\varphi_j(t))^{p^{j+1}}$ and $\varphi_j(t^p)^{p^j}$ are congruent $\bmod p^{j+1}$. Consequently,

$$T_{k-1}\big((\varphi_0(t))^p, \dots, (\varphi_{k-1}(t))^p\big) \equiv T_{k-1}(\varphi_0(t^p), \dots, \varphi_{k-1}(t^p)) \mod p^k.$$

But $P_{k-1}(t) = T_{k-1}(\varphi_0(t), \dots, \varphi_{k-1}(t))$ by definition, and so $P_{k-1}(t^p) = T_{k-1}(\varphi_0(t^p), \dots, \varphi_{k-1}(t^p))$. The lemma now follows from the congruence $P_k(t) \equiv P_{k-1}(t^p) \bmod p^k$. $\square$

2.4 COROLLARY. *For each n prime to p and $k \geq 0$ there exists $a_{n,k} \in H_{2np^k}(BU, R)$ defined inductively by $d_{e_{np^k}} = T_k(a_{n,0}, \ldots, a_{n,k})$.*

PROOF. Let S_n denote the polynomial of 2.2. By 2.3 we must show that

$$S_{np^k}(d_1, \ldots, d_{np^k}) - S_{np^{k-1}}(d_1, \ldots, d_{np^{k-1}}) \qquad (*)$$

vanishes mod p^k. For any α of weight $w(\alpha) = np^k$ with some entry prime to p, the coefficient of d^α in $(*)$ is $np^k\{\alpha\}/|\alpha|$. We must check that $\nu(\{\alpha\}) \geq \nu(|\alpha|)$ where $\nu(\alpha)$ is the exponent of p in a prime power decomposition. But $\{\alpha\} = \{\alpha_1, \alpha_2 + \cdots + \alpha_k\} \cdot \{\alpha_2, \ldots, \alpha_k\}$, so this follows from the familiar $\nu(\{\beta\}) \geq \nu(|\beta|)$ where $\beta = (\beta_1, \beta_2)$ with β_1 prime to p.

In fact, the inequality $\nu(\{p^m\beta\}) \geq \nu(|\beta|)$ holds for any $\beta = (\beta_1, \ldots, \beta_s)$. As before, it suffices to show this for $s = 2$ with β_1 prime to p. Let $\pi(i)$ denote the product of all natural numbers $< i$ which are prime to p. By some simple bookkeeping we obtain $\{p\beta\}/\{\beta\} = \pi(p\beta_1 + p\beta_2)/\pi(p\beta_1)\pi(p\beta_2)$ and hence

$$\{p^m\beta\} \prod_{i=1}^{m} \pi(p^i\beta_1)\pi(p^i\beta_2) = \{\beta\} \prod_{i=1}^{m} \pi(p^i(\beta_1 + \beta_2)).$$

But $\nu(\{\beta\}) \geq \nu(|\beta|)$ from above, while $\nu(\{p^m\beta\}) = \nu(\{\beta\})$ since π takes values prime to p.

Returning to the polynomial $(*)$, if $\alpha = p\beta$ then by 2.2 the coefficient of d^α is

$$\pm\left(\frac{np^k}{|p\beta|}\{p\beta\} - \frac{np^{k-1}}{|\beta|}\{\beta\}\right) = \pm\frac{np^{k-1}}{|\beta|}(\{p\beta\} - \{\beta\}).$$

But since $\{p\beta\}/\{\beta\} = \pi(|p\beta|)/\pi(p\beta_1) \cdots \pi(p\beta_{np^k})$ it follows that

$$\begin{aligned}\pi(p\beta_1) \cdots \pi(p\beta_{np^k})(\{p\beta\} - \{\beta\}) \\ = \{\beta\}(\pi(|p\beta|) - \pi(p\beta_1) \cdots \pi(p\beta_{np^k})).\end{aligned}$$

If $q = \min(\nu(\beta_1), \ldots, \nu(\beta_{np^k}))$, then $\nu(\{\beta\}) \geq \nu(|\beta|) - q$ by the paragraph above. By Wilson's Theorem (see e.g. [20]), $\pi(|p\beta|) \equiv (-1)^{|\beta|/p^q} \bmod p^{q+1}$ and $\pi(p\beta_i) \equiv (-1)^{|\beta_i|/p^q} \bmod p^{q+1}$. Thus $\pi(|p\beta|) - \pi(p\beta_1) \cdots \pi(p\beta_{np^k}) \equiv 0 \bmod p^{q+1}$ so that $p^{\nu(|\beta|)+1}$ divides $\{p\beta\} - \{\beta\}$, and the coefficients of d^α in $(*)$ vanishes mod p^k. $\square$

Using 2.2 it follows that $a_{n,k} = nd_{np^k}$ + decomposables and thus $\{a_{n,k}|n$ is prime to p, $k \geq 0\}$ is a polynomial basis of $H_*(BU, R)$. For each n prime to p let A_n denote the subalgebra of $H_*(BU, R)$ generated by $a_{n,0}, a_{n,1}, \ldots$. Since A_n is pure in $H_*(BU, \mathbf{Z}_{(p)})$ (if $mx \in A_n$ for some integer m then $x \in A_n$) and $d_{e_{np^k}}$ is primitive, it follows from the defining relation for $a_{n,k}$ that A_n is a Hopf subalgebra for $R = \mathbf{Z}_{(p)}$ and hence for any p-local R by universal coefficients. This proves the first part of the following result which was first established by Husemoller using different methods [11].

2.5 THEOREM. *There are isomorphisms of Hopf algebras*

$$H_*(BU,R) \cong \bigotimes_{n \text{ prime to } p} A_n \quad \text{and}$$

$$H_*(\Omega^{2k}W,R) \cong \bigotimes_{\substack{n \text{ prime to } p\\ n+k\equiv 0 \operatorname{mod}(p-1)}} A_n.$$

When $R = \mathbf{Z}_{(p)}$ this decomposition is best possible: given Hopf subalgebras $B, C \subseteq H_*(BU, \mathbf{Z}_{(p)})$ such that $H_*(BU, \mathbf{Z}_{(p)}) = B \otimes C$, then for any n prime to p either $A_n \subseteq B$ or $A_n \subseteq C$.

PROOF. We prove the last statement first. Since B and C are pure, it follows from [17, 6.16] that any primitive $d_{e_{np^j}}$ is in B or in C. Suppose inductively that $d_{e_n} = a_{n,0}, a_{n,1}, \ldots, a_{n,j-1} \in B$. By 2.4, $x = d_{e_{np^j}} - T_j((a_{n,0})^p, \ldots, (a_{n,j-1})^p)$ is p-divisible and thus so is its image $[x]$ in $H_*(BU, \mathbf{Z}_{(p)})//B \cong C$. But if $d_{e_{np^j}} \in C$, $[x] = d_{e_{np^j}}$ which is not p-divisible by 2.2. Thus $d_{e_{np^j}}$ and hence $a_{n,j}$ and A_n lie in B.

Because the Hurewicz map sends some generator $\omega_m \in \pi_{2m}(BU) = \mathbf{Z}_{(p)}$ to $(m-1)!d_{e_m}$, it follows from 1.1 that $H_*(\Omega^{2k}W, \mathbf{Z}_{(p)})$ is the smallest pure Hopf subalgebra of $H_*(BU, \mathbf{Z}_{(p)})$ containing the primitives $d_{e_{np^j}}$ where $n + k \equiv 0 \operatorname{mod}(p-1)$. Then from the argument above we have

$$H_*(\Omega^{2k}W,R) = \bigotimes_{\substack{n \text{ prime to } p\\ n+k\equiv 0 \operatorname{mod}(p-1)}} A_n$$

for $R = \mathbf{Z}_{(p)}$ and hence for any p-local R by universal coefficients. □

By the same arguments we have a decomposition in cohomology.

2.6 THEOREM. *For n prime to p and $j \geq 0$ define $a^*_{n,j} \in H^{2np^j}(BU, R)$ inductively by $c_{e_{np^j}} = T_k(a^*_{n,0}, \ldots, a^*_{n,j})$. Let A^*_n denote the polynomial Hopf subalgebra $R[a^*_{n,0}, a^*_{n,1}, \ldots]$. Then there exist isomorphisms of Hopf algebras*

$$H^*(BU,R) \cong \bigotimes_{n \text{ prime to } p} A^*_n \quad \text{and}$$

$$H^*(\Omega^{2k}W,R) \cong \bigotimes_{\substack{n \text{ prime to } p\\ n+k\equiv 0 \operatorname{mod}(p-1)}} A^*_n.$$

The homology and cohomology decompositions of BU (and of $\Omega^{2k}W$) are dual. If we give $H_*(BU, R)$ the basis of monomials in $\{a_{n,j}\}$, then $c_{e_{np^k}}/n$ is dual to $a_{n,k}$. The subalgebras A_n and A^*_n are in some sense the minimal bipolynomial Hopf algebras [21].

3. The homology of H-maps. We can now describe the induced homomorphism of an H-map $f\colon BU \to BU$. Since the Hurewicz map induces an isomorphism between $\pi_*(BU) \otimes \mathbf{Q}$ and the primitives of $H_*(BU, \mathbf{Q})$ ([5], [17]), the characteristic sequence λ of f may be defined by $f_*(d_{e_m}) = \lambda_m d_{e_m}$

where d_{e_m} is as in 2.2. Thus for each n prime to p and $k \geqslant 0$ we may apply f_* to the defining relation 2.4 to obtain

$$\lambda_{np^k} T_k(a_{n,0}, \ldots, a_{n,k}) = T_k\big(f_*(a_{n,0}), \ldots, f_*(a_{n,k})\big). \tag{3.1}$$

We illustrate the use of this equation in the ring $H_*(BU, \mathbf{Z}_{(p)})$ where the $a_{n,k}$ are not p-divisible. Suppose $\nu(\lambda_{np^k}) = j > 0$. Then $(f_*(a_{n,0}))^{p^k}$ is divisible by p and hence by p^{p^k}. If $j > 1$ divides both sides of 3.1 by p, conclude that $(f_*(a_{n,1}))^{p^{k-1}}$ is divisible by $p^{p^{k-1}}$, and continue inductively. It follows that $f_*(a_{n,0}), \ldots, f_*(a_{n,\min(j-1,k)})$ all vanish mod p and, if $(j-1) < k$, that $f_*(a_{n,j}) \not\equiv 0$.

Restating this in terms of surplus (1.5), we have that $s(f)_{np^k} > 0$ iff $f_*(a_{n,0}), \ldots, f_*(a_{n,k})$ all vanish mod p. Moreover, if $s(f)_{np^k} \leqslant 0$ and $j = k + s(f)_{np^k}$, then $f_*(a_{n,0}), \ldots, f_*(a_{n,j-1})$ all vanish mod p while $f_*(a_{n,j})$ does not. In particular, if $f_*(a_{n,k})$ is the first nonvanishing (mod p) term of $f_*(a_{n,0})$, $f_*(a_{n,1}), \ldots$, then we must have $s(f)_{np^k} = 0$, $s(f)_{np^{k+1}} = -1, \ldots$. This proves Assertion 1.6.

3.2 Definition. For any H-map f: $BU \to BU$ and n prime to p define the nth index of f, denoted $\delta_n(f) = \delta_n$, by $\delta_n = j$ if $s(f)_{np^j} = 0$ and $\delta_n = \infty$ if no such j exists.

By 1.7 the index δ_n is constant as n varies within a residue class mod$(p-1)$. The discussion above may be summarized by saying that $f_*(a_{n,j})$ vanishes mod p for $j < \delta_n$ while $f_*(a_{n,\delta_n}) \not\equiv 0$ mod p.

3.3 Theorem. *For each n prime to p and $k \leqslant \delta_n$ we have*

$$f_*(a_{n,k}) = \big(\lambda_{np^k}/p^k\big) T_k(a_{n,0}, \ldots, a_{n,k}) + px$$

for some x in the ideal in A_n generated by all $f_*(a_{n,i})$, $i < k$. If f satisfies the growth condition $s(f)_{np^i} < ps(f)_{np^{i-1}} - 1$ for $i \leqslant k$, then $\nu(px) > \nu(\lambda_{np^k}/p^k) = s(f)_{np^k}$ and thus $\nu(f_*(a_{n,k})) = s(f)_{np^k}$.

For the final statement we assume that the ground ring R is $\mathbf{Z}_{(p)}$. For $x \in H_*(BU, R)$ the symbol $\nu(x)$ then denotes the maximal j such that $x = p^j y$ for some $y \in H_*(BU, R)$. The growth condition is not satisfied in general (examples can be easily constructed using 1.4) but is satisfied by a number of maps f of geometric interest. The last statement often permits an explicit description of the torsion in the $\mathbf{Z}_{(p)}$ cohomology of the fiber of f.

Proof. If we solve the defining equation 3.1 for $f_*(a_{n,k})$ we obtain

$$\big(\lambda_{np^k}/p^k\big) T_k(a_{n,0}, \ldots, a_{n,k}) - \big(1/p^k\big) T_{k-1}\Big(\big(f_*(a_{n,0})\big)^p, \ldots, \big(f_*(a_{n,k-1})\big)^p\Big).$$

But $(f_*(a_{n,i}))^{p^{k-i}}/p^{k-i} = ((f_*(a_{n,i}))^{p^{k-i}-1}/p^{k-i}) f_*(a_{n,i})$ when $i < k$ where the first factor is evidently p-divisible since $f_*(a_{n,i})$ is. Thus the second polynomial above has the desired form px.

To prove the second assertion we may suppose inductively that $\nu(f_*(a_{n,i})) = s(f)_{np^i}$ for $i < k$ and thus

$$\nu\Big(\big(f_*(a_{n,i})\big)^{p^{k-i}}/p^{k-i}\Big) = p^{k-i}s(f)_{np^i} - (k-i).$$

But the growth condition implies that $ps(f)_{np^{k-1}} - 1 < p^2s(f)_{np^{k-2}} - 2 < \dots$ and hence

$$\nu(px) = \nu\Big(\big(f_*(a_{n,k-1})\big)^p/p\Big) = ps(f)_{np^{k-1}} - 1 > s(f)_{np^k}. \quad \square$$

Evaluating $f_*(a_{n,k})$ for $k > \delta_n$ is more difficult since λ_{np^k} and $f_*(a_{n,i})$ have insufficient p-divisibility to carry out the manipulations above. A less direct approach seems more fruitful. Fix some n prime to p with finite δ_n and for each $k > \delta_n$ let $u_k = \lambda_{np^k}/p^{\delta_n}$, a unit in $\mathbf{Z}_{(p)}$. By 1.7 it follows that $(1/u_k) \equiv (1/u_{k-1}) \bmod p^{k-\delta_n}$. Then using 2.3 we may inductively define polynomials $\rho_{k-\delta_n}(t) = \rho_{k-\delta_n}(t_0, \dots, t_{k-\delta_n})$ and $\xi_{k-\delta_n}(t) = \xi_{k-\delta_n}(t_1, \dots, t_k)$ for $k \geqslant \delta_n$ as follows.

$$(1/u_k)T_{k-\delta_n}(t) = T_{k-\delta_n}\big(\rho_0(t), \dots, \rho_{k-\delta_n}(t)\big),$$
$$-p^{k-\delta_n+1}T_{\delta_n-1}\big(t_{k-\delta_n+1}, \dots, t_k\big) = T_{k-\delta_n}\big(\xi_0(t), \dots, \xi_{k-\delta_n}(t)\big). \quad (3.4)$$

It follows immediately that $\xi_{k-\delta_n}$ is always p-divisible and that $\rho_{k-\delta_n}(t) = (1/u_k)t_k$ + decomposables. Hence the following result includes a partial description of $f_*(a_{n,k})$.

3.5 THEOREM. *If n is prime to p and $k \geqslant \delta_n$ then*

$$a_{n,k-\delta_n}^{p^{\delta_n}} = \rho_{k-\delta_n}\big(f_*(a_{n,\delta_n}), \dots, f_*(a_{n,k})\big) + \xi_{k-\delta_n}(a_{n,1}, \dots, a_{n,k}) + px$$

for some x in the ideal in A_n generated by all $f_*(a_{n,i})$, $i < k$.

PROOF. For the proof we abbreviate $a_{n,i} = a_i$, $\xi_{i-\delta_n}(a_1, \dots, a_i) = \xi_{i-\delta_n}$, and $\rho_{i-\delta_n}(f_*(a_{\delta_n}), \dots, f_*(a_i)) = \rho_{i-\delta_n}$. For $k = \delta_n$ the result is established in 3.3, so suppose inductively that 3.5 holds for all i such that $\delta_n \leqslant i < k$. Then using the relations $a_i^{p^{\delta_n}} = \rho_{i-\delta_n} + \xi_{i-\delta_n} + px_i$, the p-divisibility of $\xi_{i-\delta_n}$, and the multinomial theorem it follows (compare the proof of 2.4) that

$$T_{k-\delta_n-1}\big(a_0^{p^{\delta_n+1}}, \dots, a_{k-\delta_n-1}^{p^{\delta_n+1}}\big) = T_{k-\delta_n-1}(\rho_0^p, \dots, \rho_{k-\delta_n-1}^p)$$
$$+ T_{k-\delta_n-1}(\xi_0^p, \dots, \xi_{k-\delta_n-1}) + p^{k-\delta_n+1}\bar{x} \quad (*)$$

for some $\bar{x}$ in the ideal generated by $f_*(a_0), \dots, f_*(a_{k-1})$. If we multiply the defining equations 3.1 by the unit $p^{\delta_n}/\lambda_k = 1/u_k$ and substitute the identities $(*)$, the left-hand side of the resulting equation is

$$p^{\delta_n}T_{k-\delta_n-1}(\rho^p) + p^{\delta_n}T_{k-\delta_n-1}(\xi^p) + p^{k+1}\bar{x}$$
$$+p^k a_{k-\delta_n}^{p^{\delta_n}} + p^{k+1}T_{\delta_n-1}(a_{k-\delta_n+1}, \dots, a_k).$$

Substituting the defining relation 3.4 for $\xi_{k-\delta_n}$ and simplifying via the identity $T_{k-\delta_n}(\xi) = T_{k-\delta_n-1}(\xi^p) + p^{k-\delta_n}\xi_{k-\delta_n}$ this becomes

$$p^{\delta_n}T_{k-\delta_n-1}(\rho^p) + p^{k+1}\bar{x} + p^k a_{k-\delta_n}^{p^{\delta_n}} - p^k\xi_{k-\delta_n}. \qquad (**)$$

Using the arguments of 3.3 it follows that the right-hand side of 3.1 (multiplied by $p^{\delta_n}/\lambda_{np^k} = 1/u_k$ as before) may be written as

$$p^{k+1}\bar{\bar{x}} + (p^{\delta_n}/u_n)T_{k-\delta_n}(f_*(a_{\delta_n}), \ldots, f_*(a_k))$$

for some $\bar{\bar{x}}$ in the ideal generated by $f_*(a_0), \ldots, f_*(a_{k-1})$. Then substituting the defining equation 3.4 for $\rho_{k-\delta_n}$ and the identity $T_{k-\delta_n}(\rho) = T_{k-\delta_n-1}(\rho^p) + p^{k-\delta_n}\rho_{k-\delta_n}$ this becomes

$$p^{k+1}\bar{\bar{x}} + p^{\delta_n}T_{k-\delta_n-1}(\rho^p) + p^k\rho_{k-\delta_n}. \qquad (***)$$

The desired result follows by equating (**) and (***). □

The above work made no use of the properties of the H-map f beyond the relation 3.1. Since the cohomology homomorphism f^*: $H^*(BU, R) \to H^*(BU, R)$ sends each primitive c_{e_n} to some multiple of itself and since $\langle c_{e_n}, d_{e_n}\rangle = \pm n$, it follows that $f^*(c_{e_n}) = \lambda_n c_{e_n}$ and 3.3 and 3.5 hold for cohomology.

3.6 Theorem. *For each n prime to p we have*

$$f^*(a_{n,k}^*) = (\lambda_{np^k}/p^k)T_k(a_{n,0}^*, \ldots, a_{n,k}^*) + px_1^* \quad \text{for } k < \delta_n,$$

$$(a_{n,k-\delta_n}^*)^{p^{\delta_n}} = \rho_{k-\delta_n}(f^*(a_{n,\delta_n}^*), \ldots, f^*(a_{n,k})) + \xi_{k-\delta_n}(a_{n,1}^*, \ldots, a_{n,k}^*) + px_2^* \quad \text{if } k \geq \delta_n$$

where x_j^* lies in the ideal in A_n^* generated by all $f^*(a_{n,i}^*)$, $i < k$, and $\xi_{k-\delta_n}$, $\rho_{k-\delta_n}$ are as in 3.4. If f satisfies the additional growth condition $s(f)_{np^i} < ps(f)_{np^{i-1}} - 1$ for $i \leq \delta_n$, then $\nu(f^*(a_{n,k}^*)) = s(f)_{np^k}$ for $k \leq \delta_n$.

For the last statement we again assume that the ground ring is $\mathbf{Z}_{(p)}$. The above result (and the homology analogue) simplifies considerably if we reduce mod p.

3.7 Corollary. *The Hopf algebra homomorphism f^*: $H^*(BU, \mathbf{Z}/p) \to H^*(BU, \mathbf{Z}/p)$ induced by an H-map f satisfies*

$$f^*(a_{n,k}^*) = \begin{cases} 0 & \text{if } k < \delta_n, \\ u_k(a_{n,k-\delta_n}^*)^{p^{\delta_n}} + x_k & \text{if } k \geq \delta_n, \end{cases}$$

where x_k lies in the ideal in A_n^* generated by all $(a_{n,j}^*)^{p^{\delta_n}}$, $j \leq k - \delta_n - 1$, and $u_k \in \mathbf{Z}_{(p)}$ is a unit.

4. The mod p **cohomology of the fiber.** For any commutative Noetherian ring R and for any simple fibration $F \to E \to B$ with E and B connected and of finite type over R there is an Eilenberg-Moore spectral sequence $\{E_r\}$ converging from $E_2 = \mathrm{Tor}_{H^*(B,R)}(R, H^*(E, R))$ to $E_\infty = E_0H^*(F, R)$, the bigraded module associated to some filtration of $H^*(F, R)$ ([**9**], [**10**]). This spectral sequence is natural for morphisms of fibrations, and the filtration on $H^*(F, R)$ behaves well with respect to products. Applying the diagonal map we obtain a spectral sequence of bigraded rings. If $E \to B$ is an H-map there is an induced multiplication on F and, when R is a field, we have a spectral sequence of bigraded Hopf algebras.

Any map f: $BU \to BU$ may be regarded as a simple fibration [**24**, §2.8] with $H^*(BU, R)$ of finite type over R if R is p-local. For such fibrations we may apply the beautiful Eilenberg-Moore spectral sequence collapse theorems of May ([**14**], [**10**]), Munkholm [**18**], and others ([**3**], [**12**], [**22**], [**27**]). The following special case of May's result [**10**, Theorem B] will suffice. We use the notation $\bigoplus M$ to denote the graded module associated to a bigraded module M.

4.1 THEOREM. *If f: $BU \to BU$ is a fibration with fiber F, then there is a natural isomorphism of bigraded algebras*

$$E_0H^*(F, R) \cong \mathrm{Tor}_{H^*(BU,R)}(R, H^*(BU, R))$$

for some natural, nonpositive, decreasing filtration of $H^*(F, R)$. If f is an H-map and R is a field then this is an isomorphism of bigraded Hopf algebras. In either case, the map in filtration zero is induced by the canonical map

$$H^*(BU, R) \otimes_{H^*(BU,R)} R \to H^*(F, R)$$

induced by inclusion of the fiber. Finally, there is no additive extension problem and we obtain an isomorphism of graded R modules

$$H^*(F, R) \cong \bigoplus \mathrm{Tor}_{H^*(BU,R)}(R, H^*(BU, R)).$$

4.2. To determine the torsion products above we first compute $\mathrm{Tor}_{A_n^*}(R, A_n^*)$ for each n prime to p by means of the classical Koszul resolution. Let σ: $H^i(BU, R) \to H^{i-1}(U, R)$ denote the cohomology suspension; this is the composite

$$H^i(BU, R) \xrightarrow{\pi^*} H^i(PBU, U; R) \xrightarrow{\delta^{-1}} H^{i-1}(U, R)$$

where $U \to PBU \to^{\pi} BU$ is the path fibration. Then $\sigma(A_n^*) \subseteq H^*(U, R)$ is an exterior Hopf subalgebra $E\{\sigma a_{n,j}^* | j \geq 0\}$ on primitive generators $\sigma a_{n,j}^* \in H^{2np^j-1}(U, R)$. The module $\sigma(A_n^*) \otimes_R A_n^*$ with differential generated by $\sigma a_{n,j}^* \to a_{n,j}^*$ is an A_n^*-free resolution of R (compare [**3**, §2]). Moreover, the Hopf algebra structures on A_n^* and $\sigma(A_n^*)$ induce one on the resolutions. Applying $\otimes_{f^*} A_n^*$ and checking the definition of product and coproduct for

Tor, it follows that $\bigoplus \mathrm{Tor}_{A_n^*}(R, A_n^*) = H(\sigma A_n^* \otimes A_n^*, d)$ as a graded ring (and as a Hopf algebra if f is an H-map and R is a field) with differentials given by $\sigma a_{nj}^* \to f^*(a_{nj}^*)$. When R is a field the external product on Tor is also an isomorphism **[18]** so we obtain a natural isomorphism

$$\bigoplus \mathrm{Tor}_{H^*(BU,R)}(R, H^*(BU, R)) \cong \bigotimes_{n \text{ prime to } p} \mathrm{Tor}_{A_n^*}(R, A_n^*).$$

4.3 THEOREM. *Let f: $BU \to BU$ be an H-map, $R = \mathbf{Z}/p$, and regard A_n^* as a module over itself via f^*. Then*

$$\bigoplus \mathrm{Tor}_{A_n^*}(\mathbf{Z}/p, A_n^*) \cong E\left\{\sigma a_{nj}^* | 0 \leqslant j < \delta_n\right\} \otimes_{\mathbf{Z}/p} \left(A_p^* // \xi^{\delta_n} A_n^*\right)$$

as Hopf algebras where ξ: $x \to x^p$ is the Frobenius map.

PROOF. By **[23**, 1.5**]** there is an isomorphism of Hopf algebras

$$\bigoplus \mathrm{Tor}_{A_n^*}(\mathbf{Z}/p, A_n^*) \cong \mathrm{Tor}_{\mathrm{subker} f^*}(\mathbf{Z}/p, \mathbf{Z}/p) \otimes (A_n^* // f^* A_n^*)$$

where subker f^* is the unique Hopf subalgebra of A_n^* generating ker f^*: $A_n^* \to A_n^*$. But by 3.7, subker $f^* = \mathbf{Z}/p[a_{n,0}^*, \ldots, a_{n,\delta_n - 1}^*]$ and $f^* A_n^* = \xi^{\delta_n} A_n^*$, and the result follows by inspection of the Koszul resolution. □

Thus by the Kunneth theorem there is an isomorphism of Hopf algebras

$$\bigoplus E_0 H^*(F, \mathbf{Z}/p) \cong \bigotimes_{n \text{ prime to } p} E\left\{\sigma a_{nj}^* | 0 \leqslant j < \delta_n\right\} \otimes \left(A_n^* // \xi^{\delta_n} A_n^*\right).$$

Comparing this with the Eilenberg-Moore spectral sequence for the universal principal fibration (see e.g. **[22]**) and applying some standard Hopf algebra arguments, we may replace the term on the left with $H^*(F, \mathbf{Z}/p)$. This proves the following.

4.4 THEOREM. *Let f: $BU \to BU$ be an H-map with fiber F. Then $H^*(F, \mathbf{Z}/p) \cong \bigotimes_{n \text{ prime to } p} E\{\sigma a_{nj}^* | 0 \leqslant j < \delta_n\} \otimes (A_n^* // \xi^{\delta_n} A_n^*)$ as Hopf algebras. The induced homomorphism of the inclusion i: $F \to B$ is given by the natural projections $A_n^* \to A_n^* // \xi^{\delta_n} A_n^*$, while σa_{nj}^* pulls back via the canonical map $U \to F$ to the cohomology suspension of a_{nj}^*.*

By 1.3 the fiber F of an H-map f: $BU \to BU$ decomposes as an H space into a product $F(0) \times F(2) \times \cdots \times F(2p-4)$ where (in the notation of 1.4) $F(2k)$ may be defined as the fiber of f_{2k}: $BU \to BU$ or of $\tilde{f}_{2k}$: $\Omega^{2k} W \to \Omega^{2k} W$. Applying 4.4 to f_{2k} we obtain the following.

4.5 COROLLARY. *If $F = F(0) \times F(2) \times \cdots \times F(2p-4)$ is the natural H space decomposition of F, then*

$$H^*(F(2k), \mathbf{Z}/p) = \bigotimes_{\substack{n \text{ prime to } p \\ n+k \equiv 0 \bmod (p-1)}} E\left\{\sigma a_{nj}^* | 0 \leqslant j < \delta_n\right\} \otimes \left(A_n^* // \xi^{\delta_n} A_n^*\right).$$

5. The Bockstein spectral sequence of the fiber. For any chain complex (C, ∂) of abelian groups the long exact sequence in homology associated to the coefficient sequence $0 \to \mathbf{Z} \to^p \mathbf{Z} \to \mathbf{Z}/p \to 0$ may be regarded as an exact couple whose underlying graded spectral sequence is the Bockstein spectral sequence. The E_1 term is $E_1(C) = H_*(C, \mathbf{Z}/p)$ and the differential for E_r is the rth order Bockstein β_r which may be described as follows. Given $\{c\} \in H_*(C, \mathbf{Z}/p)$ where $c \in C$ is a chain satisfying $\partial(c) = p^r c'$, define $\beta_r\{c\} = \{c'\} \in H_*(C, \mathbf{Z}/p)$. This spectral sequence is discussed in considerable detail in [6] and [7]. We recall some useful facts.

5.1. If x generates a direct summand $\mathbf{Z}/p^r$ in $H_*(C)$, then $j_{(r)}(y) \neq 0$ in $E_r(C)$ where $j_{(r)}$: $H_*(C) \to E_r(C)$ is the map induced by j: $H_*(C) \to H_*(C, \mathbf{Z}/p)$.

5.2. Kernel$(j_r) = pH_*(C) + T_{r-1}$ where T_{r-1} is the subgroup of $H(C)$ annihilated by p^{r-1}.

5.3. $E_\infty(C) = (H_*(C)/\text{Torsion}) \otimes \mathbf{Z}/p$.

5.4. Given complexes C and D, the canonical map $H_*(C, \mathbf{Z}/p) \otimes H_*(D, \mathbf{Z}/p) \to H_*(C \otimes D, \mathbf{Z}/p)$ induces an isomorphism of chain complexes $E_r(C) \otimes E_r(D) \to E_r(C \otimes D)$ for all $r \geqslant 1$.

Now suppose f: $BU \to BU$ is an H-map with fiber F. Using 4.1 and 4.2 with $R = \mathbf{Z}_{(p)}$ it follows that $H^*(F, \mathbf{Z}_{(p)})$ is the homology of the chain complex $\otimes_{n \text{ prime to } p} \sigma(A_n^*) \otimes A_n^*$ with degree one differential defined by $\partial(\sigma(a_{n,j}^*)) = f^*(a_{n,j}^*)$. But since

$$E_r\Big(\bigotimes_{n \text{ prime to } p} (\sigma(A_n^*) \otimes A_n^*)\Big) = \bigotimes_{n \text{ prime to } p} E_r(\sigma(A_n^*) \otimes A_n^*)$$

by 5.4, to describe the Bockstein spectral sequence of the space F it will suffice to describe that of the component complexes $\sigma(A_n^*) \otimes A_n^*$. Of particular interest for geometric applications is a two-step complex of surplus s. This is a chain complex $\sigma(A_n^*) \otimes A_n^*$ in which the H-map f inducing the differential as in 4.2 satisfies $s(f)_{np^k} = s > 0$ for all $k < \delta_n$.

5.5 THEOREM. *Let $\sigma(A_n^*) \otimes A_n^*$ be a two-step complex of surplus s. For $r \leqslant s$ the Bockstein spectral sequence is given by*

$$E_r \cong E\left\{\sigma a_{n,0}^*, \ldots, \sigma a_{n,\delta_n - 1}^*\right\} \otimes A_n^*//\xi^{\delta_n} A_n^* \otimes \mathbf{Z}/p.$$

If $r \geqslant s$ we have

$$E_r \cong E\left\{\sigma a_{n,r-s}^*, \ldots, \sigma a_{n,r-s+\delta_n-1}^*\right\} \otimes A_n^*(r-s)//\xi^{\delta_n} A_n^*(r-s) \otimes \mathbf{Z}/p$$

where $A_n^*(k) = \mathbf{Z}_{(p)}[a_{n,k}^*, a_{n,k+1}^*, \ldots]$.

PROOF. For convenience we may first suppose that the characteristic subsequence $\lambda_n, \lambda_{pn}, \ldots$ used to define the differential on $\sigma(A_n^*) \otimes A_n^*$ is given by $p^s, p^{s+1}, \ldots, p^{s+\delta_n-1}, p^{\delta_n}, p^{\delta_n} \ldots$. By 3.6 we may choose elements

$c_k = \sigma a^*_{n,k} - \sum_{i<k}\sigma(a^*_{n,i}) \otimes x_i$ for suitable $x_i \in A^*_n$ such that $\partial(c_k) = p^s T_k(a^*_{n,0}, \ldots, a^*_{n,k})$ if $k < \delta_n$ and $\partial(c_k) = (a^*_{n,k-\delta_n})^{p^{\delta_n}} - \xi_{k-\delta_n}(a^*_{n,1}, \ldots, a^*_{n,k})$ for $k \geq \delta_n$. Then setting $b_0 = c_0$, $b_k = c_k - b_0(a^*_{n,0})^{p^{k-1}} - \cdots - b_{k-1}(a^*_{n,k-1})^{p-1}$ for $0 < k < \delta_n$, and $b_k = c_k$ for $k \geq \delta_n$, it follows that we may rewrite $\sigma(A^*_n) \otimes A^*_n$ as a complex $C(0) = E\{b_0, b_1, \ldots\} \otimes A^*_n$ with differential $\partial_{0,s}$ defined by $\partial_{0,s}(b_k) = p^{k+s}a^*_{n,k}$ for $k < \delta_n$ and $\partial_{0,s}(b_k) = (a_{n,k-\delta_n})^{p^{\delta_n}} - \xi_{k-\delta_n}(a^*_{n,1}, \ldots, a^*_{n,k})$ if $k \geq \delta_n$.

For each $k \geq 0$ let $C(k)$ denote the free $\mathbf{Z}_{(p)}$ module $E\{b_k, b_{k+1}, \ldots\} \otimes A^*_n(k)$ where b_j is, as above, an exterior generator of degree $np^j - 1$. On $C(0)$ we have already defined differentials $\partial_{0,s}$ for each $s > 0$. We may similarly define $\partial_{k,s}$ on $C(k)$ by

$$\partial_{k,s}(b_j) = p^{j-k+s}a^*_{n,j} \quad \text{if } k \leq j < k + \delta_n,$$

$$\partial_{k,s}(b_j) = \left(a^*_{n,j-\delta_n}\right)^{p^{\delta_n}} - \xi_{j-k-\delta_n}\left(a^*_{n,k+1}, \ldots, a^*_{n,j}\right) \quad \text{if } j \geq k + \delta_n.$$

Assertion 1. $E_r(C(0), p^i\partial_{0,s-i}) \cong E_r(C(0), p^{i-1}\partial_{0,s-i+1})$ for all $i = 1, \ldots, s - 1$ and $r > i$.

Assertion 2. $E_r(C(k), p^k\partial_{k,1}) \cong E_r(C(k+1), p^{k+1}\partial_{k+1,1})$ if $r > k + 1$.

Assuming these for the moment the proof proceeds as follows. Clearly $E_i(C(k), p^i\partial_{k,s}) = C(k)$ with ith Bockstein the mod p reduction of $\partial_{k,s}$. Thus

$$E_{i+1}\left(C(k), p^i\partial_{k,s}\right) = E\left\{b_k, \ldots, b_{k+\delta_n-1}\right\} \otimes A^*_n(k)//\xi^{\delta_n}A^*_n(k) \otimes \mathbf{Z}/p$$

by the argument of 4.3. From the definition of the elements b_j this is isomorphic to $E(\sigma a^*_{n,k}, \ldots, \sigma a^*_{n,k+\delta_n-1}) \otimes A^*_n(k)//\xi^{\delta_n}A^*_n(k) \otimes \mathbf{Z}/p$. Thus if $r \leq s$ it follows by Assertion 1 that

$$\begin{aligned} E_r(C(0), \partial_{0,s}) &\cong E_r\left(C(0), p^{r-1}\partial_{0,s-r+1}\right) \\ &\cong E\left\{\sigma a^*_{n,0}, \ldots, \sigma a^*_{n,\delta_n-1}\right\} \otimes A^*_n//\xi^{\delta_n}A^*_n \otimes \mathbf{Z}/p. \end{aligned}$$

When $r \geq s$ it follows by Assertions 1 and 2 that

$$\begin{aligned} E_r(C(0), \partial_{0,s}) &\cong E_r\left(C(0), p^{s-1}\partial_{0,1}\right) \\ &\cong E_{r-s+1}\left(C(0), \partial_{0,1}\right) \\ &\cong E_{r-s+1}\left(C(r-s), p^{r-s}\partial_{r-s,1}\right) \\ &\cong E\left\{\sigma a_{n,r-s}, \ldots, \sigma a_{n,r-s+\delta_n-1}\right\} \otimes A^*_n(r-s)//\xi^{\delta_n}A^*_n(r-s) \otimes \mathbf{Z}/p. \end{aligned}$$

To prove Assertion 1 it suffices to check that $E_r(C(0), p\partial_{0,s-i}) \cong E_r(C(0), \partial_{0,s-i+1})$ for $r > 1$ and $i = 1, \ldots, s - 1$. Define a chain map φ: $(C(0), \partial_{0,s-i+1}) \to (C(0), p\partial_{0,s-i})$ by setting $\varphi = 1$ on A^*_n, $\varphi(b_i) = b_i$ if $i < \delta_n$, and $\varphi(b_i) = pb_i$ if $i \geq \delta_n$. Then φ induces an isomorphism on E_2. For as noted above,

$$E_2(C(0), p\partial_{0,s-i}) = E_1(C(0), \partial_{0,s-i})$$
$$= E\{b_0, \ldots, b_{\delta_n-1}\} \otimes A_n^*//\xi^{\delta_n}A_n^* \otimes \mathbf{Z}/p.$$

But by the same argument we have $E_1(C(0), \partial_{0,s-i+1}) = E\{b_0, \ldots, b_{\delta_n-1}\} \otimes A_n^*//\xi^{\delta_n}A_n^* \otimes \mathbf{Z}/p$ with the entire homology group lying in the kernel of the first Bockstein.

The second assertion is proved similarly. We will define a chain map φ: $(C(k+1), p\partial_{k+1,1}) \to (C(k), \partial_{k,1})$ which induces an isomorphism on E_2. Let $\varphi|_{A(k+1)}$ be the canonical inclusion $A(k+1) \subseteq A(k)$, define $\varphi(b_j) = b_j$ for $k+1 \leqslant j < k+\delta_n$, and

$$\varphi(b_{k+\delta_n}) = pb_{k+\delta_n} - b_k \otimes (a_{n,k}^*)^{p^{\delta_n}-1} - \cdots - b_{k+\delta_n-1} \otimes (a_{n,k+\delta_n-1}^*)^{p-1}.$$

Next abbreviate $\xi_j(a_{n,k+1}^*, \ldots, a_{n,k+j+\delta_n}^*) = \xi$ and $\xi_{j-1}(a_{n,k+2}^*, \ldots, a_{n,k+j+\delta_n}^*) = \hat{\xi}_{j-1}$, and note that the defining relation 3.4 yields

$$pT_{k-\delta_n-1}(\hat{\xi}_0, \ldots, \hat{\xi}_{k-\delta_n-1}) = T_{k-\delta_n}(\xi_0, \ldots, \xi_{k-\delta_n}).$$

Then using the p-divisibility of ξ_0 and the binomial theorem (compare the proof of 2.4) it follows inductively that $\hat{\xi}_{j-1} = \xi_j + \eta_j\xi_0$ for some $\eta_j \in A_n^*(k)$. We complete the definition of φ by setting $\varphi(b_{k+\delta_n+j}) = pb_{k+\delta_n+j} + \eta_j x$ for all $j > 0$ where $x = pb_{k+\delta_n} - b_k(a_{n,k}^*)^{p^{\delta_n}-1}$ satisfies $\partial_{k,1}(x) = p\xi_0$.

To compute φ_* on E_2 recall first that

$$E_2(C(k+1), p\partial_{k+1,1})$$
$$= E\{b_{k+1}, \ldots, b_{k+\delta_n}\} \otimes A_n^*(k+1)//\xi^{\delta_n}A_n^*(k+1) \otimes \mathbf{Z}/p$$

and

$$E_1(C(k), \partial_{k,1}) = E\{b_k, \ldots, b_{k+\delta_n-1}\} \otimes A_n^*(k)//\xi^{\delta_n}A_n^*(k) \otimes \mathbf{Z}/p$$

as before. The first Bockstein on the latter ring is defined by the conditions $\beta_1(b_k) = a_k$ and $\beta_1(b_{k+1}) = \cdots = \beta_1(b_{k+\delta_n-1}) = \beta_1(A_n^*(k)//\xi^{\delta_n}A_n^*(k)) = 0$. It follows that Ker β_1 is the subring generated by $b_{k+1}, \ldots, b_{k+\delta_n-1}, b_k \otimes (a_{n,k}^*)^{p^{\delta_n}-1}$, and $A_n^*(k)$, while $\mathrm{Im}(\beta_1) = a_{n,k}^* \cdot (\mathrm{Ker}\ \beta_1)$. By a change of basis we may thus write Ker $\beta_1 = E\{b'_{k+1}, \ldots, b'_{k+\delta_n}\} \otimes A_n^*(k)//\xi^{\delta_n}A_n^*(k) \otimes \mathbf{Z}/p$ where $b'_j = b_j$ if $j < k+\delta_n$ and $b'_{k+\delta_n} = pb_{k+\delta_n} - b_k \otimes (a_{n,k}^*)^{p^{\delta_n}-1} - \cdots - b_{k+\delta_n-1} \otimes (a_{n,k+\delta_n-1}^*)^{p-1}$. Thus $E_2(C(k), \partial_{k,1}) = E\{b'_{k+1}, \ldots, b'_{k+\delta_n}\} \otimes (A_n^*(k+1)//\xi^{\delta_n}A_n^*(k+1))$ with φ inducing an isomorphism on E_2. □

The Bockstein spectral sequence for the complex $C = \sigma(A_n^*) \otimes A_n^*$ with arbitrary surplus $s_n, s_{np}, \ldots, s_{np^{\delta_n-1}}$ is, unfortunately, still a mess. If $s = \min\{s_n, \ldots, s_{np^{\delta_n-1}}\}$ and $C^1 = \sigma(A_n^*) \otimes A_n^*$ is the complex of constant surplus s obtained by dividing the characteristic sequence for C by appropriate powers of p, then applying 5.1, 5.2, and 5.3 to the computation 5.5 yields reasonably explicit information about the torsion group $H(C^1)$. In particular cases one can then profitably compare $H(C)$ and $H(C^1)$ by studying, via the mapping cone construction, the homology of the chain map $\varphi\colon C \to C^1$ defined by

$$\varphi(\sigma a_{n,j}^*) = p^{s_{np^j} - s}\sigma a_{n,j}^* \quad \text{for } j < \delta_n,$$
$$\varphi(\sigma a_{n,j}^*) = \sigma a_{n,j}^* \quad \text{for } j \geqslant \delta_n,$$

and $\varphi|_{A_n^*} = 1$. Such considerations are omitted here in the hope the general case will soon be brought under control.

6. An important special case. The $\mathbf{Z}_{(p)}$ cohomology of the fiber of an H-map $f\colon BU \to BU$ with index never exceeding 1 admits a direct description. Such an explicit computation, though less transparent than the Bockstein spectral sequence computation of the previous section, will be necessary in a subsequent study of the classifying space of smoothing theory.

We first suppose that $f\colon BU \to BU$ is an H-map with characteristic sequence λ such that, for a fixed n prime to p, $\delta_n(f) = 1$ and $\lambda_n = p^m w$ for some unit $w \in \mathbf{Z}_{(p)}$. Regarding A_n^* as a module over itself via f^* as usual, we describe $\bigoplus \mathrm{Tor}_{A_n^*}(\mathbf{Z}_{(p)}, A_n^*)$ as both a $\mathbf{Z}_{(p)}$ module and an algebra.

6.1 THEOREM. *There is an isomorphism of graded algebras*

$$\bigoplus \mathrm{Tor}_{A_n^*}(\mathbf{Z}_{(p)}, A_n^*) \cong A_n^*//f^*A_n^* = A_n^*/I$$

where I is the A_n^* ideal generated by

$$\lambda_n a_{n,0}^*,\ (a_{n,0}^*)^p - pw_1 a_{n,1}^*,\ (a_{n,1}^*)^p - pw_2 a_{n,2}^*, \ldots$$

for certain units $w_1, w_2, \ldots \in \mathbf{Z}_{(p)}$. The submodule of elements of degree $2r$ is isomorphic to $\bigoplus_\alpha \mathbf{Z}/\mathrm{ind}(\alpha)$ where $\alpha = (\alpha_1, \alpha_2, \ldots)$ ranges over all sequences of nonnegative integers of weight r each of whose nonzero entries is $< (p-1)$ and lies in the subsequence $\alpha_n, \alpha_{np}, \alpha_{np^2}, \ldots$, and where $\mathrm{ind}(\alpha) = p^{m+j}$ if α_{np^j} is the first nonzero entry of α.

PROOF. By the definition of torsion product (compare e.g. [3]) we have that $\mathrm{Tor}^{0,*}_{A_n^*}(\mathbf{Z}_{(p)}, A_n^*) = \mathbf{Z}_{(p)} \otimes_{f^*} A_n^* = A_n^*//f^*A_n^*$ where the quotient module $A_n^*//f^*A_n^*$ is torsion because f^* is a rational isomorphism. But since $(A_n^*//f^*A_n^*) \otimes \mathbf{Z}/p \cong A_n^* \otimes \mathbf{Z}/p//f^*A_n^* \otimes \mathbf{Z}/p$ is a vector space with one generator in each dimension nj by 3.7, it follows by 4.4 and inspection that

$$\begin{aligned}\mathrm{Tor}_{A_n^* \otimes \mathbf{Z}/p}(\mathbf{Z}/p, A_n^* \otimes \mathbf{Z}/p) &\cong E\{\sigma a_{n,0}^*\} \otimes A_n^*//\xi A_n^* \otimes \mathbf{Z}/p \\ &\cong (A_n^*//f^*A_n^* \otimes \mathbf{Z}/p) \oplus \mathrm{Tor}_1((A_n^*//f^*A_n^*), \mathbf{Z}/p).\end{aligned}$$

But by universal coefficients $\mathrm{Tor}_{A_n^* \otimes \mathbf{Z}/p}(\mathbf{Z}/p, A_n^* \otimes \mathbf{Z}/p)$ is isomorphic to

$(\mathrm{Tor}_{A_n^*}(\mathbf{Z}_{(p)}, A_n^*) \otimes \mathbf{Z}/p) \oplus \mathrm{Tor}_1(\mathrm{Tor}_{A_n^*}(\mathbf{Z}_{(p)}, A_n^*), \mathbf{Z}/p)$. Thus $\mathrm{Tor}^{s,t}_{A_n^*}(\mathbf{Z}_{(p)}, A_n^*) = 0$ if $s \neq 0$ and hence

$$\mathrm{Tor}_{A_n^*}(\mathbf{Z}_{(p)}, A_n^*) = \bigoplus \mathrm{Tor}_{A_n^*}(\mathbf{Z}_{(p)}, A_n^*) = A_n^*//f^*A_n^*.$$

For each $j \geq 0$ let I_j denote the A_n^* ideal generated by $f^*(a_{n,0}^*), \ldots, f^*(a_{n,j}^*)$ and assume inductively that I_{j-1} is generated by $\lambda_n a_{n,0}^*$, $(a_{n,0}^*)^p - w_1 p a_{n,1}^*, \ldots, (a_{n,j-2}^*)^p - w_{j-1} p a_{n,j-1}^*$ for some units $w_1, \ldots, w_{j-1} \in \mathbf{Z}_{(p)}$. It follows from 3.4 and 3.5 that $f^*(a_{n,j}^*) = v_j((a_{n,j-1}^*)^p - \xi_{j-1}(a_{n,1}^*, \ldots, a_{n,j}^*)) + x$ where $x \in I_{j-1}$ and $v_j = p/\lambda_{np^j}$, a unit in $\mathbf{Z}_{(p)}$. By definition ξ_{j-1} is a p-divisible polynomial given by $\xi_{j-1}(a_{n,1}^*, \ldots, a_{n,j}^*) = -pa_{n,j}^* - p^{1-j}T_{j-2}(\xi_0^p, \ldots, \xi_{j-2}^p)$. If $j = 1$ the last term on the right vanishes, and if $j = 2$ this becomes $\xi_1(a_{n,1}^*, a_{n,2}^*) = -pa_{n,2}^* + p^{p-1}(a_{n,1}^*)^p$. In general, since ξ_{j-1} is homogeneous of degree np^j and since the degrees of $a_{n,2}^*, \ldots, a_{n,j}^*$ are multiples of np^2, we may regard ξ_{j-1} as a polynomial in $(a_{n,1}^*)^p, a_{n,2}^*, \ldots, a_{n,j}^*$. Thus $\xi_{j-1} \equiv -pa_{n,j}^* + p\xi'(a_{n,2}^*, \ldots, a_{n,j-1}^*) \bmod I_{j-1}$ for some polynomial ξ'. As before, regard ξ' as a polynomial in $(a_{n,2}^*)^p, a_{n,3}^*, \ldots, a_{n,j}^*$ and continue inductively. Thus we may write

$$\xi_{j-1}(a_{n,1}^*, \ldots, a_{n,j}^*) \equiv -pa_{n,j}^* + p\eta(a_{n,j-1}^*)^p \mod I_{j-1}$$

for some $\eta \in \mathbf{Z}_{(p)}$. It follows that I_j is generated over A_n^* by I_{j-1} and $(a_{n,j-1}^*)^p - w_k p a_{n,j}^*$ where $w_k = 1/(\eta p - 1)$ and the first part of 6.1 is established.

It follows from the work above that any monomial in A_n^* is congruent modulo $I = I_1 \cup I_2 \cup \ldots$ to some multiple of a monomial in "reduced form"

$$\left((a_n^*)^{(k_0, \ldots, k_j)} = (a_{n,0}^*)^{k_0} \cdots (a_{n,j}^*)^{k_j} \quad \text{where } k_i < p \text{ for all } i\right).$$

Moreover this monomial is unique since distinct reduced monomials have different degrees. We must show that $(a_n^*)^{(0, \ldots, 0, k_i, \ldots, k_j)}$ has order precisely p^{m+i} if $k_i \neq 0$. From the congruences

$$p^i(a_n^*)^{(0, \ldots, 0, k_i, \ldots, k_j)} \equiv (p^{i-1}/w_i)(a_n^*)^{(0, \ldots, 0, p, k_i - 1, \ldots, k_j)} \equiv \ldots$$
$$\equiv (1/w_1 \cdots w_i)(a_n^*)^{(p, p-1, \ldots, p-1, k_i - 1, k_{i+1}, \ldots, k_j)}$$

mod I it will suffice to show that any monomial $(a_n^*)^{(k_0, \ldots, k_j)}$ with $0 < k_0 \leq p$ and $k_i < p$ for $i > 0$ has order p^m in A_n^*/I. Thus suppose

$$\eta(a_n^*)^{(k_0, \ldots, k_j)} = \lambda_n a_{n,0}^* x_0 + \sum_{i=0}^{l} \left((a_{n,i}^*)^p - w_i p a_{n,i}^*\right) x_i \qquad (*)$$

for some $x_0, \ldots, x_l \in A_n^*$ and $\eta \in \mathbf{Z}_{(p)}$, and consider the evaluation map E: $A_n^* \to \mathbf{Z}_{(p)}$ sending

$$a_{n,k}^* \to \left(p^{p^k - p^{k-1} - \cdots - p - 1}\right) / \left(w_1^{p^{k-1}} w_2^{p^{k-2}} \cdots w_{k-1}^p w_k\right).$$

This map is defined so that $E((a^*_{n,k-1})^p - w_k p a^*_{n,k}) = 0$ for all $k > 0$. This implies that among all monomials $(a^*_n)^{(l_0, \ldots, l_j)}$ of a given degree the one whose E image has the lowest p divisibility is the monomial in reduced form. Applying E to $(*)$ above we obtain $\eta E((a^*_n)^{(k_0, \ldots, k_j)}) = \lambda_n E(a^*_{n,0}x_0)$. Thus if $k_0 < p$, η must be divisible by at least p^m. If $k_0 = p$, then both $(a^*_n)^{(k_0, \ldots, k_j)}$ and $a^*_{n,0}x_0$ have a factor $(a^*_{n,0})^p$, so again p^m divides η. □

Using universal coefficients we can now combine the calculation above for various values of n to prove Theorem D of the introduction.

Proof of Theorem D. For each finite subset $S' \subseteq S$ let $C_{S'}$ denote the complex $\bigotimes_{n\in S'} \sigma A^*_n \otimes A^*_n$ with differential defined via f^* as in 4.2, and let $T_{S'} \subseteq T$ consist of those sequences α whose nonzero entries lie in some subsequence $\alpha_n, \alpha_{pn}, \ldots$ for $n \in S'$. We will show by induction on the size of S' that

$$H_m(C_{S'}) \cong \bigoplus_{j=1}^{\infty} \bigoplus_{i=0}^{j-1} \bigoplus_{\substack{\alpha\in T_{S'} \\ l(\alpha)=j \\ 2w(\alpha)=m+i}} (\mathbf{Z}/\operatorname{ind}\alpha)^{\binom{j-1}{i}}. \tag{6.2}$$

This proves 6.2 since, for each $m > 0$, $H^m(F, \mathbf{Z}_{(p)}) = H_m(C_{S'})$ for some finite $S' \subseteq S$.

If S' consists of a single element, then $l(\alpha) = 1$ for any $\alpha \in T_{S'}$ and the formula 6.2 reduces to that of 6.1. Thus suppose $S' \subseteq S$ is some finite set for which (6.2) holds and choose $n \in S \setminus S'$. By the Künneth formula

$$\begin{aligned} H_m(C_{S'\cup\{n\}}) &= H_m(C_{S'} \otimes C_{\{n\}}) \\ &\cong \underbrace{\bigoplus_{m'+m'=m} H_{m'}(C_{S'}) \otimes H_{m''}(C_{\{n\}})}_{\text{(a)}} \\ &\quad \oplus \underbrace{\bigoplus_{\substack{m'+m''=m+1 \\ m'\neq 0\neq m'}} H_{m'}(C_{S'}) \otimes H_{m''}(C_{\{n\}})}_{\text{(b)}}. \end{aligned}$$

This uses the fact that the reduced groups $\tilde{H}_*(C_{S'})$ and $\tilde{H}_*(C_{\{n\}})$ are torsion. But for any $\alpha \in T_{S'\cup\{n\}}$ there are isomorphisms

$$\begin{aligned} \mathbf{Z}/\operatorname{ind}\alpha &\cong \mathbf{Z}/\min(\operatorname{ind}\alpha', \operatorname{ind}\alpha'') \\ &\cong \mathbf{Z}/\operatorname{ind}\alpha' \otimes \mathbf{Z}/\operatorname{ind}\alpha'' \end{aligned}$$

where $\alpha = \alpha' + \alpha''$ for some unique $\alpha' \in T_S$, $\alpha'' \in T_{\{n\}}$.

Then applying the inductive assumption, the subgroup of (a) above corresponding to $m' = 0$ or $m'' = 0$ is given by

$$\bigoplus_{j=1}^{\infty} \bigoplus_{i=0}^{j-1} \bigoplus_{\substack{\alpha \in T_{S' \cup \{n\}},\ \alpha'=0 \text{ or } \alpha''=0 \\ l(\alpha)=j \\ 2w(\alpha)=m+i}} (\mathbf{Z}/\text{ind } \alpha)^{\binom{j-1}{i}}. \qquad (a_1)$$

When $\alpha'' \neq 0$ we have $l(\alpha') = l(\alpha) - 1$ so that the remaining contribution from (a) is

$$\bigoplus_{j=1}^{\infty} \bigoplus_{i=0}^{j-2} \bigoplus_{\substack{\alpha \in T_{S' \cup \{n\}},\ \alpha' \neq 0 \neq \alpha'' \\ l(\alpha)=j \\ 2w(\alpha)=m+i}} (\mathbf{Z}/\text{ind } \alpha)^{\binom{j-2}{i}}. \qquad (a_2)$$

Similarly, applying the inductive assumption to (b) yields

$$\bigoplus_{j=1}^{\infty} \bigoplus_{i=0}^{j-2} \bigoplus_{\substack{\alpha \in T_{S' \cup \{n\}},\ \alpha' \neq 0 \neq \alpha'' \\ l(\alpha)=j \\ 2w(\alpha)=m+i+1}} (\mathbf{Z}/\text{ind } \alpha)^{\binom{j-2}{i}}$$

$$= \bigoplus_{j=1}^{\infty} \bigoplus_{i=1}^{j-1} \bigoplus_{\substack{\alpha \in T_{S' \cup \{n\}},\ \alpha' \neq 0 \neq \alpha'' \\ l(\alpha)=j \\ w(\alpha)=m+i}} (\mathbf{Z}/\text{ind } \alpha)^{\binom{j-2}{i-1}}. \qquad (b_1)$$

Then adding (a_1), (a_2), and (b_1) and substituting the identity

$$\binom{j-2}{i} + \binom{j-2}{i-1} = \binom{j-1}{i}$$

we obtain the desired formula 6.2 for the set $S' \cup \{n\}$. □

We conclude with a remark about nontorsion elements. Most of the results in this paper involve an H-map f: $BU \to BU$ whose characteristic sequence has only nonzero entries. Equivalently, the reduced $\mathbf{Z}_{(p)}$ cohomology of the fiber of f is torsion. If λ has trivial entries, however, there is a reasonably simple algorithm for computing the rank of the cohomology. For any set S of positive integers and for any positive integers k and n, denote by $p_S(k, n)$ the number of distinct ways of writing n as a sum of integers from the set S (an element may be used more than once) in which exactly k different elements from S are used. For example, if S is the set of all positive integers then $p_S(1, n) + p_S(2, n) + \dots$ is the number of partitions of n. Let $p_S(k, n) = 0$ if n is not an integer.

6.4 THEOREM. *Let F: $BU \to BU$ be an H-map with characteristic sequence λ and fiber F, and write S for the set of all indices n such that $\lambda_n = 0$. Then for every $m > 0$ the $\mathbf{Z}_{(p)}$ rank of $H^m(F, \mathbf{Z}_{(p)})$ is given by*

$$\sum_{i,j>0} \binom{i+j}{i} p_S(i+j, (i+m)/2).$$

We omit the proof which is a straightforward counting argument involving the rational Eilenberg-Moore spectral sequence of f.

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Department of Mathematics, State University of New York at Albany, Albany, New York 12222